

# On the range of self-interacting random walks on an interval

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## Abstract

We consider the ranges of a one-parameter family of self-interacting walks up to the time of exit from an interval. We derive the weak convergence of the appropriately scaled range. We show that the distribution functions of the limits satisfy a certain class of de Rham's functional equations. We examine the regularity of them.

## 1 Introduction

The range of random walk has been studied for a long time. Examining the range at the time the random walk leaves an interval is a simple and natural concern. Recently, Athreya, Sethuraman and Tóth [1] considered questions of this kind. They studied the range, local times and periodicity or “parity” statistics of some nearest-neighbor Markov random walks up to the time of exit from an interval of  $N$  sites. They derived several associated scaling limits as  $N \rightarrow \infty$  and some of them connect with the entropy of an exit distribution, generalized Ray-Knight constructions, and Bessel and Ornstein-Uhlenbeck square processes, among other objects.

Inspired by [1], we consider the ranges of a certain class of self-interacting random walks up to the time of exit from an interval. The study of self-interacting walks originated from the modeling of polymer chains in chemical physics. There are various models in this study. We treat the model defined by Denker and Hattori [2], Hambly, Hattori and Hattori [4], Hattori and Hattori [5], [6]. They constructed a natural one-parameter family of self-repelling and self-attracting walks on  $\mathbb{Z}$  and the infinite pre-Sierpiński

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gasket. It interpolates between self-avoiding walk and simple random walk continuously in the sense of exponents.

In general, most of the studies of self-interacting walks, even if they are one-dimensional, are difficult due to the lack of Markov property. In the studies of Markov walks, we can use techniques in Analysis, especially, potential theory. However, in the case of non-Markov walks, we cannot use most of the techniques used in the studies of Markov walks. Most of the arguments in [1] depend heavily on the Markov property. Therefore, we have to use alternative methods for our study.

Now we state our settings and results briefly. Let  $W_\infty$  be the path space of the nearest-neighbor walk starting at 0 on  $\mathbb{Z}$ . Let  $\{P^u\}_{u \geq 0}$  be a one-parameter family of probability measures on  $W_\infty$  defined by [2] and [5]. We will give precise definitions of them in Section 2. We remark that  $P^0$  defines the self-avoiding walk on  $\mathbb{Z}$  and  $P^1$  defines the standard simple random walk.

**Definition 1.1.** Let  $n \in \mathbb{N} = \{1, 2, \dots\}$  and  $\omega \in W_\infty$ . Let  $R_n(\omega)$  be the range of  $\omega$  up to the time of exit from  $\{-2^n, \dots, 2^n\}$ , that is,

$$R_n(\omega) = (\text{the number of points which } \omega \text{ visits before it hits the points } \{\pm 2^n\}).$$

Then, we have the following results which are analogous to [1], Proposition 2.1.

**Theorem 1.2.** (1) Let  $u \geq 0$ . Then,  $\tilde{P}_n^u = P^u \circ ((R_n/2^n) - 1)^{-1}$  converges weakly to a probability measure  $\tilde{P}^u$  on  $[0, 1]$ ,  $n \rightarrow \infty$ .

(2) Let  $u > 0$ . Then the distribution function of  $\tilde{P}^u$  satisfies a certain class of de Rham's functional equations [3] :

$$f(x) = \begin{cases} \Phi(A_{u,0}; f(2x)) & 0 \leq x \leq 1/2 \\ \Phi(A_{u,1}; f(2x-1)) & 1/2 \leq x \leq 1, \end{cases} \quad (1.1)$$

where we let  $\Phi(A; z) = \frac{az+b}{cz+d}$  for  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and

$$A_{u,0} = \begin{pmatrix} x_u & 0 \\ -u^2 x_u^2 & 1 \end{pmatrix}, \quad A_{u,1} = \begin{pmatrix} 0 & x_u \\ -u^2 x_u^2 & 1 - u^2 x_u^2 \end{pmatrix}, \quad \text{where } x_u = \frac{2}{1 + \sqrt{1 + 8u^2}}.$$

(3) If  $u = 1$ ,  $\tilde{P}^u$  is absolutely continuous. If  $u \neq 1$ ,  $\tilde{P}^u$  is singular.

We remark that  $\tilde{P}^0 = \tilde{P}_n^0 = \delta_{\{0\}}$ , where  $\delta$  denotes a point mass.

Let us denote the Hausdorff dimension of  $K \subset [0, 1]$  by  $\dim_H(K)$ . Let us define the Hausdorff dimension of a probability measure  $\mu$  on  $[0, 1]$  by

$\dim_H \mu = \inf\{\dim_H(K) : K \in \mathcal{B}([0, 1]), \mu(K) = 1\}$ . Let  $s(p) = -p \log p - (1-p) \log(1-p)$  for  $p \in [0, 1]$ .

The followings are direct consequences of the author [7]. For the proof we refer the reader to [7].

**Theorem 1.3.** (1) If  $u \neq 1$  and  $0 < u < \sqrt{3}$ , then  $\dim_H \tilde{P}^u < 1$ .  
(2) If  $0 < u < 1$ , then  $\dim_H \tilde{P}^u \leq s(x_u)/\log 2$ . Moreover,  $\tilde{P}^u(K) = 0$  for any Borel set  $K$  with  $\dim_H(K) < s(2x_u/(1+x_u))/\log 2$ .

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## 2 Preliminaries

Most of the definitions and propositions in this section are stated in [2] and [5]. See them for detail.

For each  $n \in \mathbb{N} \cup \{0\}$ , let

$$W(n) = \{(\omega(0), \omega(1) \dots \omega(n)) \in \mathbb{Z}^{n+1} : \omega(0) = 0, |\omega(i) - \omega(i+1)| = 1, 0 \leq i \leq n-1\}.$$

Let  $W^* = \bigcup_{n=0}^{\infty} W(n)$ . Let  $L(\omega) = n$  for  $\omega \in W(n)$ .

For  $\omega \in W^*$ , we define  $T_i^M(\omega)$ ,  $i, M \in \mathbb{N} \cup \{0\}$ , by  $T_0^M(\omega) = 0$ ,  $T_i^M(\omega) = \min\{j > T_{i-1}^M(\omega) : \omega(j) \in 2^M \mathbb{Z} \setminus \{\omega(T_{i-1}^M(\omega))\}\}$ ,  $i \geq 1$ . Let  $T_i^M(\omega) = +\infty$  if the above minimum does not exist.

We define a map  $Q_M : W^* \rightarrow W^*$ ,  $M \in \mathbb{N}$ , by  $(Q_M \omega)(i) = \omega(T_i^M(\omega))$  for  $i$  such that  $T_i^M(\omega) < +\infty$ . Let  $Q_0$  be the identity map on  $W^*$ .

Let  $(2^{-M} Q_M \omega)(i) = 2^{-M} \omega(T_i^M(\omega))$ . Then,  $2^{-M} Q_M \omega \in W^*$  and  $L(2^{-M} Q_M \omega) = k$ , where  $k = \max\{i : T_i^M(\omega) < \infty\}$ .

Let  $W_{N,+}(\text{resp. } -) = \{\omega \in W^* : L(\omega) = T_1^N(\omega), \omega(T_1^N(\omega)) = +(\text{resp. } -)2^N\}$  and  $W_N = W_{N,+} \cup W_{N,-}$ .

We define a map  $G(N, n) : W_{N+n,+} \rightarrow \bigcup_{k=1}^{\infty} (W_{n,+} \cap \{L(\cdot) = k\}) \times W_{N,+}^k$ ,  $N, n \in \mathbb{N}$ , in the following manner :

- (1) For  $\omega \in W_{N+n,+}$ , let  $\omega' = 2^{-N} Q_N \omega$ . Then,  $\omega' \in W_{n,+}$ .
- (2) Let  $1 \leq j \leq L(\omega')$ . Let  $\omega_j = (0, \omega(T_{j-1}^N(\omega) + 1) - \omega(T_{j-1}^N(\omega)), \dots, \omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega))) \in W_N$ .
- (3) Let  $\tilde{\omega}_j = \text{sign}(\omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega))) \omega_j \in W_{N,+}$ .
- (4) Let  $G(N, n)(\omega) = (\omega', \tilde{\omega}_1, \dots, \tilde{\omega}_{L(\omega')})$ .

We see that  $G(N, n)$  is bijective.

Now we will define a probability measure  $P_{N,\pm}^u$ ,  $u \geq 0$ , on  $W_{N,\pm}$  by induction in the following manner. We recall that  $x_u = 2/(1 + \sqrt{1 + 8u^2})$ .

- (1) Let  $P_{1,+}^u(\{\omega\}) = u^{L(\omega)-2}x_u^{L(\omega)-1}$ ,  $\omega \in W_{1,+}$ , where we adopt the convention  $0^0 = 1$  and  $0^n = 0$ ,  $n \geq 1$ .  
(2) Let  $\omega \in W_{N+1,+}$  and  $(\omega', \tilde{\omega}_1, \dots, \tilde{\omega}_{L(\omega')}) = G(N, 1)(\omega)$ . Then, let

$$P_{N+1,+}^u(\{\omega\}) = P_{1,+}^u(\{\omega'\}) \prod_{i=1}^{L(\omega')} P_{N,+}^u(\{\tilde{\omega}_i\}).$$

- (3) We define  $P_{N,-}^u(\{\omega\}) = P_{N,+}^u(\{-\omega\})$  for  $\omega \in W_{N,-}$ ,  $N \in \mathbb{N}$ .

Let  $P_N^u$  be a probability measure on  $W_N$  given by  $P_N^u = (P_{N,+}^u + P_{N,-}^u)/2$ .

**Proposition 2.1** ([2], Proposition 2.2). *Let  $\omega \in W_{N+n,+}$ ,  $N, n \in \mathbb{N}$ .*

*Let  $(\omega', \tilde{\omega}_1, \dots, \tilde{\omega}_{L(\omega')}) = G(N, n)(\omega)$ . Then,*

$$P_{N+n,+}^u(\{\omega\}) = P_{n,+}^u(\{\omega'\}) \prod_{i=1}^{L(\omega')} P_{N,+}^u(\{\tilde{\omega}_i\}).$$

Hence,  $P_{n,+}^u = P_{N+n,+}^u \circ (2^{-N}Q_N)^{-1}$  on  $W_{n,+}$ .

We denote the set of the paths of infinite length by

$$W_\infty = \{(\omega(0), \omega(1), \dots) \in \mathbb{Z}^{\mathbb{N} \cup \{0\}} : \omega(0) = 0, |\omega(i) - \omega(i+1)| = 1, i \geq 0\}.$$

Let the  $\sigma$ -algebra on this set be the family of subsets which is generated by cylinder sets.

**Proposition 2.2** ([2], Proposition 2.5). *There exists a probability measure  $P^u$  on  $W_\infty$  such that*

$$P^u(\{\omega \in W_\infty : \omega(j) = \tilde{\omega}(j), 0 \leq j \leq L(\tilde{\omega})\}) = \frac{1}{2} P_{N,+(resp.-)}^u(\{\tilde{\omega}\}),$$

for  $\tilde{\omega} \in W_{N,+(resp.-)}$ ,  $N \geq 1$ .

### 3 Range of random walk on $\{-2^n, \dots, +2^n\}$ and its scaling limit

Here and henceforth, we assume that  $u > 0$ . We define  $R_n(\omega)$  for  $\omega \in W_{N,+}$ ,  $N \geq n$ , as in Definition 1.1.

We remark that  $P^u(R_n = 2^n + k) = P_{n,+}^u(R_n = 2^n + k)$ ,  $0 \leq k \leq 2^n$ ,  $n \geq 1$ .

**Lemma 3.1.**

$$P_{N,+}^u \left( \frac{R_N}{2^N} - 1 \geq \frac{k}{2^n} \right) = P_{n,+}^u \left( \frac{R_n}{2^n} - 1 \geq \frac{k}{2^n} \right),$$

for any  $N \geq n$ ,  $0 \leq k \leq 2^n$  and  $n \geq 1$ .

*Proof.* Let  $N > n$ . Then,

$$\begin{aligned} P_{N,+}^u \left( \frac{R_N}{2^N} - 1 \geq \frac{k}{2^n} \right) &= P_{N,+}^u \left( \{\omega \in W_{N,+} : \omega \text{ hits the point } \{-2^{N-n}k\}\} \right) \\ &= P_{N,+}^u \left( \{\omega : Q_{N-n}\omega \text{ hits the point } \{-2^{N-n}k\}\} \right) \\ &= P_{N,+}^u \left( \{\omega : 2^{-(N-n)}Q_{N-n}\omega \text{ hits the point } \{-k\}\} \right) \\ &= P_{n,+}^u \left( \{\zeta \in W_{n,+} : \zeta \text{ hits the point } \{-k\}\} \right) \\ &= P_{n,+}^u \left( \frac{R_n}{2^n} - 1 \geq \frac{k}{2^n} \right), \end{aligned}$$

where in the fourth equality we have used Proposition 2.1.  $\square$

Let  $D$  be the set of dyadic rationals on  $[0, 1]$ .

**Definition 3.2.** (1) Let  $g_u$  be a function on  $D$  given by  $g_u(k/2^n) = P_{n,+}^u(R_n < 2^n + k)$ ,  $0 \leq k \leq 2^n$ . By Lemma 3.1, this is well-defined. We immediately see that  $g_u(x)$  is increasing and  $g_u(0) = 0, g_u(1) = 1$ .

(2) Let  $\tilde{g}_u$  be a function on  $[0, 1]$  given by  $\tilde{g}_u(x) = \lim_{y \in D, y > x, y \rightarrow x} g_u(y)$ ,  $0 \leq x < 1$  and  $\tilde{g}_u(1) = 1$ . This is right continuous.

**Proposition 3.3.** *The function  $g_u$  satisfies (1.1) on  $D$ , that is,*

$$P_{n+1,+}^u(R_{n+1} \leq 2^{n+1} + k) = \begin{cases} \Phi(A_{u,0}; P_{n,+}^u(R_n \leq 2^n + k)) & -1 \leq k \leq 2^n - 1 \\ \Phi(A_{u,1}; P_{n,+}^u(R_n \leq k)) & 2^n - 1 \leq k \leq 2^{n+1} - 1. \end{cases}$$

*Proof.* If  $k = -1$ , we have that  $\Phi(A_{u,0}; P_{n,+}^u(R_n \leq 2^n + k)) = \Phi(A_{u,0}; 0) = 0 = P_{n+1,+}^u(R_{n+1} \leq 2^{n+1} + k)$ . If  $k = 2^n - 1$ , we have that  $\Phi(A_{u,0}; P_{n,+}^u(R_n \leq 2^n + k)) = \Phi(A_{u,0}; 1) = \Phi(A_{u,1}; 0) = \Phi(A_{u,1}; P_{n,+}^u(R_n \leq k))$ . Then, it is sufficient to show this assertion in the following two cases.

Case 1.  $0 \leq k \leq 2^n - 1$ .

For  $\omega \in W_{n+1,+}$ , let  $(\omega', \tilde{\omega}_1, \dots, \tilde{\omega}_{L(\omega')}) = G(n, 1)(\omega)$ . Then,

$$P_{n+1,+}^u(R_{n+1} \leq 2^{n+1} + k) = \sum_{m=1}^{\infty} P_{n+1,+}^u \left( \{\omega : L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\} \right).$$

Since  $0 \leq k \leq 2^n - 1$ , we see that  $\omega' \in W_{1,+}$  does not hit  $-1$  for any  $\omega \in W_{n+1,+}$  with  $R_{n+1}(\omega) \leq 2^{n+1} + k$ . Then we see that

$$\begin{aligned} & \{\omega : L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\} \\ &= \{\omega : \omega' = (0, 1, 0, 1, \dots, 0, 1, 2), L(\omega') = 2m, R_n(\tilde{\omega}_{2i-1}) \leq 2^n + k, 1 \leq i \leq m\}. \end{aligned}$$

By the definition of  $P_{n+1,+}^u$  and  $P_{1,+}^u$ , we see that

$$\begin{aligned} & P_{n+1,+}^u(\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}) \\ &= P_{1,+}^u(\{\zeta : \zeta = (0, 1, 0, 1, \dots, 0, 1, 2), L(\zeta) = 2m\}) \cdot P_{n,+}^u(R_n \leq 2^n + k)^m. \\ &= u^{2m-2} x_u^{2m-1} P_{n,+}^u(R_n \leq 2^n + k)^m. \end{aligned}$$

Then,

$$\begin{aligned} P_{n+1,+}^u(R_{n+1} \leq 2^{n+1} + k) &= \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} P_{n,+}^u(R_n \leq 2^n + k)^m \\ &= \Phi(A_{u,0}; P_{n,+}^u(R_n \leq 2^n + k)), \end{aligned}$$

which is the desired result.

Case 2.  $2^n \leq k \leq 2^{n+1} - 1$ .

For  $\omega \in W_{n+1,+}$ , let  $(\omega', \tilde{\omega}_1, \dots, \tilde{\omega}_{L(\omega')}) = G(n, 1)(\omega)$ . If  $L(\omega') = 2m$ , we can write  $\omega' = (0, \epsilon_1, 0, \epsilon_2, \dots, 0, \epsilon_{m-1}, 0, 1, 2)$ ,  $\epsilon_i \in \{\pm 1\}$ ,  $1 \leq i \leq m-1$ . Then we see that

$$\begin{aligned} & \{\omega : L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\} \\ &= \bigcup_{i=0}^{m-1} \{\omega : \sharp(j : \epsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}. \end{aligned}$$

We remark that the union in the above is disjoint.

For  $1 \leq i \leq m-1$ ,

$$\begin{aligned} & \{\omega : \sharp(j : \epsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\} \\ &= \bigcup_{1 \leq n_1 < n_2 < \dots < n_i \leq m-1} \{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \dots < n_i\}, \\ & \quad L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}. \end{aligned}$$

We remark that the union in the above equality is disjoint.

By the definition of  $G(n, 1)$ , we see that

$$\begin{aligned} \{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\} \\ = \{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ L(\omega') = 2m, R_n(\tilde{\omega}_{2n_j}) \leq k, 1 \leq j \leq i\}. \end{aligned}$$

Then, by the definition of  $P_{n+1,+}^u$  and  $P_{1,+}^u$ ,

$$\begin{aligned} P_{n+1,+}^u(\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ L(\omega') = 2m, R_n(\tilde{\omega}_{2n_j}) \leq k, 1 \leq j \leq i\}) \\ = P_{1,+}^u(\{\omega' : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, L(\omega') = 2m\}) P_{n,+}^u(R_n \leq k)^i \\ = u^{2m-2} x_u^{2m-1} (P_{n,+}^u(R_n \leq k))^i. \end{aligned}$$

Since the number of choices  $\{n_1 < n_2 < \cdots < n_i\} \subset \{1, \dots, m-1\}$  is equal to  $\binom{m-1}{i}$ , we see that

$$\begin{aligned} P_{n+1,+}^u(\{\omega : \sharp(j : \epsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}) \\ = \sum_{1 \leq n_1 < n_2 < \cdots < n_i \leq m-1} P_{n+1,+}^u(\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}) \\ = \binom{m-1}{i} u^{2m-2} x_u^{2m-1} (P_{n,+}^u(R_n \leq k))^i, \quad 1 \leq i \leq m-1. \end{aligned}$$

This is also true for  $i = 0$ . By summing up over  $i$ , we see that

$$P_{n+1,+}^u(\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \leq 2^{n+1} + k\}) = u^{2m-2} x_u^{2m-1} (1 + P_{n,+}^u(R_n \leq k))^{m-1}.$$

Then, by summing up over  $m$ , we see that

$$\begin{aligned} P_{n+1,+}^u(R_{n+1} \leq 2^{n+1} + k) &= \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} (1 + P_{n,+}^u(R_n \leq k))^{m-1} \\ &= \Phi(A_{u,1}; P_{n,+}^u(R_n \leq k)). \end{aligned}$$

This completes the proof.  $\square$

We define some notation. Let  $X_n(x) = \lfloor 2^n x \rfloor - 2 \lfloor 2^{n-1} x \rfloor$  and  $\zeta_n(x) = \sum_{k=1}^n 2^{-k} X_k(x)$ ,  $x \in [0, 1]$ ,  $n \geq 1$ . Then,  $\zeta_n(x) \leq x < \zeta_n(x) + 2^{-n}$ ,  $x \in [0, 1]$ ,  $n \geq 1$ . Let  $\gamma_u = 1/\Phi(A_{u,0}; 1)$ . Let  $p_{u,0}(z) = (z+1)/(z+\gamma_u)$  and  $p_{u,1}(z) = 1 - p_{u,0}(z)$  for  $z > -\gamma_u$ . Let

$$\begin{pmatrix} p_{u,n}(x) & q_{u,n}(x) \\ r_{u,n}(x) & s_{u,n}(x) \end{pmatrix} = A_{u,X_1(x)} \cdots A_{u,X_n(x)}, \quad x \in [0, 1], n \geq 1.$$

**Proposition 3.4.** (1)  $g_u(\zeta_m(x)) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 0)$  and  $g_u(\zeta_m(x) + 2^{-m}) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 1)$ ,  $x \in [0, 1]$ ,  $m \geq 1$ .  
(2)  $\tilde{g}_u = g_u$  on  $D$ .  
(3)  $\tilde{g}_u$  satisfies the equation (1.1) on  $[0, 1]$ .

*Proof.* (1) We can show this assertion in the same manner as in the proof of [7], Lemma 2.1(1).

(2) By the definition of  $g_u$  and  $\tilde{g}_u$ , we have that  $\tilde{g}_u(1) = 1 = g_u(1)$ . Let  $x \in D \cap [0, 1)$ . Then, there exists  $N$  such that  $X_n(x) = 0$ ,  $n > N$ .

Then, by the assertion (1),

$$\begin{aligned} \lim_{l \rightarrow \infty} g_u(x + 2^{-l}) &= \lim_{l \rightarrow \infty} g_u(\zeta_l(x) + 2^{-l}) \\ &= \lim_{m \rightarrow \infty} \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)). \end{aligned}$$

Since  $\Phi(A_{u,0}; \cdot)$  is a contraction map on  $[0, 1]$ ,  $\lim_{m \rightarrow \infty} \Phi(A_{u,0}^m; 1) = 0$ . Then, by the assertion (1),

$$\lim_{m \rightarrow \infty} \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; 0) = g_u(x).$$

Thus we obtain the assertion (2).

(3) Since  $\tilde{g}_u(1) = 1$  and  $\Phi(A_{u,1}; 1) = 1$ , (1.1) holds for  $x = 1$ .

Let  $x \in [0, 1/2)$ . Then there exists a sequence  $\{x_n\}_n \subset D \cap [0, 1/2)$  such that  $x_n \downarrow x$ . By Proposition 3.3 and the assertion (2),  $\tilde{g}_u(x_n) = \Phi(A_{u,0}; \tilde{g}_u(2x_n))$ ,  $n \geq 1$ . Since  $\Phi(A_{u,0}; \cdot)$  is continuous and  $\tilde{g}_u$  is right continuous, we have that  $\tilde{g}_u(x) = \Phi(A_{u,0}; \tilde{g}_u(2x))$ .

In the same manner, we see that  $\tilde{g}_u(x) = \Phi(A_{u,1}; \tilde{g}_u(2x - 1))$ ,  $x \in [1/2, 1)$ . Thus we obtain the assertion (3).  $\square$

Now we show Theorem 1.2. We recall  $\tilde{P}_n^u = P^u \circ ((R_n/2^n) - 1)^{-1}$ . Let  $\tilde{P}^u$  be the probability measure on  $[0, 1]$  whose distribution function is  $\tilde{g}_u$  and satisfying  $\tilde{P}^u(\{0\}) = 0$ .

Let  $f$  be a continuous function on  $[0, 1]$  and  $\epsilon > 0$ . Then, there exists a  $m \in \mathbb{N}$  such that  $\max_{1 \leq k \leq 2^m} |f(k/2^m) - f((k-1)/2^m)| < \epsilon$ . Then, we see



that

$$\left| \int_{[0,1]} f(x) \tilde{P}_n^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) \right| < \epsilon, \quad (3.1)$$

and,

$$\left| \int_{[0,1]} f(x) \tilde{P}^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) \right| < \epsilon, \quad (3.2)$$

where we have used  $\tilde{P}_n^u(\{1\}) = P^u(R_n = 2^{n+1}) = P_{n,+}^u(R_n = 2^{n+1}) = 0$  for the first inequality, and,  $\tilde{P}^u(\{0\}) = 0$  for the second.

Let  $n > m$ . Then, by Lemma 3.1, we see that for  $1 \leq k \leq 2^m$ ,

$$\tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \tilde{P}_m^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

By Proposition 3.4(2), we see that for  $1 \leq k \leq 2^m$ ,

$$\tilde{P}^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) = \tilde{g}_u\left(\frac{k}{2^m}\right) - \tilde{g}_u\left(\frac{k-1}{2^m}\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

Then, we see that

$$\sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right).$$

Noting that (3.1) and (3.2), we see that for  $n > m$ ,

$$\left| \int_{[0,1]} f(x) \tilde{P}_n^u(dx) - \int_{[0,1]} f(x) \tilde{P}^u(dx) \right| < 2\epsilon.$$

Then we have that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f(x) \tilde{P}_n^u(dx) = \int_{[0,1]} f(x) \tilde{P}^u(dx).$$

Thus we obtain the assertion (1).

Now we immediately obtain the assertion (2) by noting the definition of  $\tilde{P}^u$  and Proposition 3.4(3).

Let  $u = 1$ . Then, the absolute continuity of  $\tilde{P}^1$  is followed from [7], Theorem 1.2(1).

**Lemma 3.5.** *Let  $u \neq 1$ . Let  $x \in [0, 1] \setminus D$ . If  $\tilde{g}_u$  is differentiable at  $x$  and  $\tilde{g}'_u(x) \in [0, +\infty)$ , then,  $\tilde{g}'_u(x) = 0$ .*

*Proof.* We assume that there exists a point  $x \in [0, 1] \setminus D$  such that  $\tilde{g}_u$  is differentiable at  $x$  and  $\tilde{g}'_u(x) \in (0, +\infty)$ .

Since  $\tilde{g}_u$  is strictly increasing and  $x \notin D$ , we have that

$$\tilde{g}'_u(x) = \lim_{n \rightarrow \infty} 2^n (\tilde{g}_u(\zeta_n(x) + 2^{-n}) - \tilde{g}_u(\zeta_n(x))) = \lim_{n \rightarrow \infty} 2^n (g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))).$$

Since  $\tilde{g}'_u(x) \in (0, +\infty)$ ,

$$\lim_{n \rightarrow \infty} \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))} = \frac{1}{2}.$$

Then, by Proposition 3.4(1),

$$p_{u, X_{n+1}(x)} \left( \frac{r_{u,n}(x)}{s_{u,n}(x)} \right) = \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))},$$

and,  $\lim_{n \rightarrow \infty} p_{u, X_{n+1}(x)}(r_{u,n}(x)/s_{u,n}(x)) = 1/2$ . Since  $p_{u,1} = 1 - p_{u,0}$ ,  $\lim_{n \rightarrow \infty} p_{u,i}(r_{u,n}(x)/s_{u,n}(x)) = 1/2$ ,  $i = 0, 1$ . Now we see that  $\lim_{n \rightarrow \infty} r_{u,n}(x)/s_{u,n}(x) = \gamma_u - 2$ . Since  $x \notin D$ , there exists infinitely many natural numbers  $n$  such that  $X_n(x) = i$ ,  $i = 0, 1$ . Since  $r_{u,n+1}(x)/s_{u,n+1}(x) = \Phi({}^t A_{u, X_{n+1}(x)}; r_{u,n}(x)/s_{u,n}(x))$ , we see that  $\Phi({}^t A_{u,i}; \gamma_u - 2) = \gamma_u - 2$ ,  $i = 0, 1$ . This is true if and only if  $u = 1$ . But this is contradict to the assumption.  $\square$

Let  $u \neq 1$ . Then, by noting Lemma 3.5 and the Lebesgue differentiation theorem, we see that  $\tilde{g}'_u = 0$  a.e. and  $\tilde{P}^u$  is singular.

These complete the proof of Theorem 1.2.

**Proposition 3.6.** (1) *Let  $u \leq \sqrt{3}$ . Then,  $\tilde{P}^u$  has no atoms.*  
(2) *Let  $u > \sqrt{3}$ . Then,  $\tilde{P}^u(\{x\}) > 0$  for any  $x \in D \cap (0, 1]$ .*

*Proof.* In this proof, let  $\Phi_{u,i}(z) = \Phi(A_{u,i}; z)$ ,  $i = 0, 1$ . Here  $f^{m+1} = f \circ f^m$ ,  $m \geq 1$ , for  $f : [0, 1] \rightarrow [0, 1]$ .

(1) If  $0 < u < \sqrt{3}$ , this assertion is easy to see. Let  $u = \sqrt{3}$ . Let  $h_i = \Phi_{\sqrt{3},i}$ ,  $i = 0, 1$ . Then we have the followings by computations.

**Lemma 3.7.** (1)  $h_0(z) < h_1(z)$  for  $z \in [0, 1]$ .  
(2)  $h'_i$ ,  $i = 0, 1$ , are strictly increasing on  $(0, 1)$ .  
(3)  $h'_0(z) \leq 3h'_1(z)$  for  $z \in (0, 1)$ .  
(4)  $h'_0(z) \leq h'_1(z)$  for  $z \geq h_1^2(0)$ .

Now it is sufficient to show the following.

**Lemma 3.8.**

$$\lim_{m \rightarrow \infty} \max_{1 \leq k \leq 2^m} \left\{ g_{\sqrt{3}} \left( \frac{k}{2^m} \right) - g_{\sqrt{3}} \left( \frac{k-1}{2^m} \right) \right\} = 0.$$

*Proof.* Let  $m \geq 3$  and  $1 \leq k \leq 2^m$ . Let  $x_i = X_i((k-1)/2^m)$ ,  $1 \leq i \leq m$ . Then,

$$\begin{aligned} g_{\sqrt{3}} \left( \frac{k}{2^m} \right) - g_{\sqrt{3}} \left( \frac{k-1}{2^m} \right) &= h_{x_1} \circ \cdots \circ h_{x_m}(1) - h_{x_1} \circ \cdots \circ h_{x_m}(0) \\ &= \int_0^1 (h_{x_1} \circ \cdots \circ h_{x_m})'(x) dx \\ &= \int_0^1 h'_{x_1}(h_{x_2} \circ \cdots \circ h_{x_m}(x)) \cdots h'_{x_{m-1}}(h_{x_m}(x)) h'_{x_m}(x) dx \\ &\leq \int_0^1 h'_{x_1}(h_1^{m-1}(x)) \cdots h'_{x_{m-1}}(h_1(x)) h'_{x_m}(x) dx \\ &\leq \int_0^1 h'_1(h_1^{m-1}(x)) \cdots 3h'_1(h_1(x)) 3h'_1(x) dx \\ &= 9 \int_0^1 (h_1^m)'(x) dx = 9(1 - h_1^m(0)), \end{aligned}$$

where we have used Proposition 3.4 (1) for the first equality, Lemma 3.7 (1) and (2) for the fourth inequality, and, Lemma 3.7 (3) and (4) for the fifth. Since  $h_1^n(0) = n/(n+1)$ ,  $n \geq 1$ , we see that  $\lim_{n \rightarrow \infty} h_1^n(0) = 1$ .  $\square$

This completes the proof of the assertion (1).

(2) Let  $x \in D \cap (0, 1)$ . Let  $x_i = X_i(x)$ ,  $i \geq 1$ . Then, there exists a unique  $m \geq 1$  such that  $x_m = 1$  and  $x_i = 0$ ,  $i \geq m+1$ . Let  $\phi = \Phi_{u, x_1} \circ \cdots \circ \Phi_{u, x_{m-1}} \circ \Phi_{u, 0}$ . Let  $n > m$  and  $y_i = X_i(x - (1/2^n))$ . Then, we have that  $y_i = x_i$ ,  $1 \leq i \leq m-1$ ,  $y_m = 0$ ,  $y_i = 1$ ,  $m+1 \leq i \leq n$ , and,  $y_i = 0$ ,  $i > n$ . By noting Proposition 3.4 (1) and  $\Phi_{u, 0}(1) = \Phi_{u, 1}(0)$ , we have that

$$g_u(x) = \phi(1), \quad g_u \left( x - \frac{1}{2^n} \right) = \phi(\Phi_{u, 1}^{n-m}(0)). \quad (3.3)$$

Since  $\Phi_{u, 1}$  is not contractive, there exists  $z_1 \in (0, 1)$  such that

$$\Phi_{u, 1}(z_1) = z_1, \quad \Phi_{u, 1}(z) > z, \quad z \in (0, z_1), \quad \Phi_{u, 1}(z) < z, \quad z \in (z_1, 1).$$

Then,  $z_1 = \lim_{n \rightarrow \infty} \Phi_{u, 1}^n(0)$  and  $\Phi_{u, 1}^n(0) \leq z_1 < 1$ ,  $n \geq 1$ .

We have that for  $n > m$ ,

$$\begin{aligned}\tilde{P}^u\left(\left(x - \frac{1}{2^n}, x\right]\right) &= g_u(x) - g_u\left(x - \frac{1}{2^n}\right) \\ &= \phi(1) - \phi(\Phi_{u,1}^{n-m}(0)) \\ &\geq \phi(1) - \phi(z_1),\end{aligned}$$

where we have used Proposition 3.4 (2) for the first equality, and, (3.3) for the second. Letting  $n \rightarrow \infty$ , we have that  $\tilde{P}^u(\{x\}) \geq \phi(1) - \phi(z_1) > 0$ .

We can show that  $\tilde{P}^u(\{1\}) > 0$  in the same manner. These complete the proof of the assertion (2).  $\square$

**Remark 3.9.** We can define  $P_{n,+}^u$  in a more general setting (see [6]). We do not know whether the main results hold or not in that case.

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